Two-Level Fractional Factorial Design

Introduction

- As the number of factors in a 2^k factorial design increases, the number of runs required for a complete replicate of the design rapidly outgrows the resources of most experimenters.
- If the experimenter can reasonably assume that certain high-order interactions are negligible, information on the main effects and low-order interactions may be obtained by running only a fraction of the complete factorial experiment.
- These fractional factorial designs are among the most widely used types of designs for product and process design, process improvement, and industrial/business experimentation.
- A major use of fractional factorials is in screening experiments—experiments in which many factors are considered and the objective is to identify those factors (if any) that have large effects.

Three key ideas behind successful fractional factorial design

- 1. The sparsity of effects principle. When there are several variables, the system or process is likely to be driven primarily by some of the main effects and low-order interactions.
- 2. The projection property. Fractional factorial designs can be projected into stronger (larger) designs in the subset of significant factors.
- 3. Sequential experimentation. It is possible to combine the runs of two (or more) fractional factorials to construct sequentially a larger design to estimate the factor effects and interactions of interest.

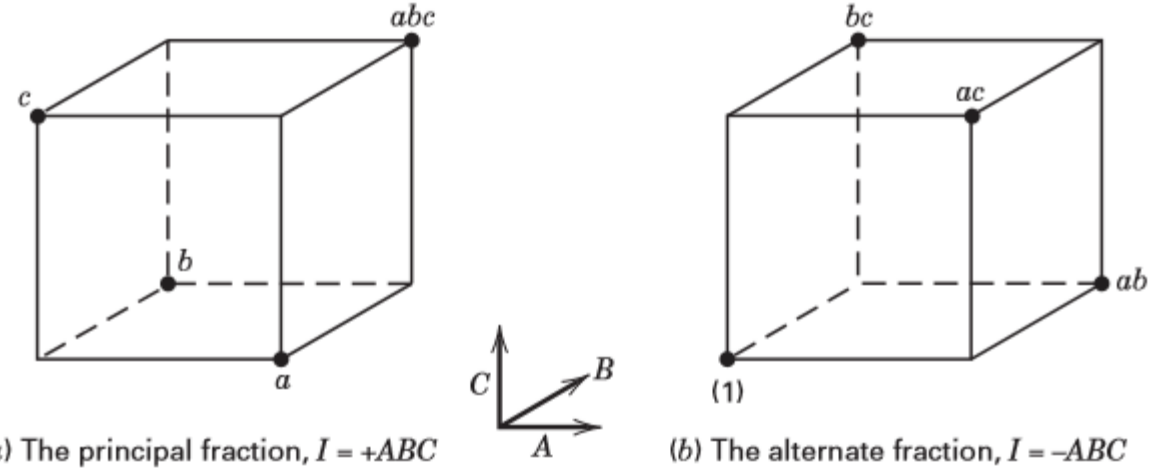
The One-Half Fraction of the 2^k Design

- Consider a situation in which three factors, each at two levels, are of interest, but the experimenters cannot afford to run all $2^3 = 8$ treatment combinations.
- They can, however, afford four runs. This suggests a one-half fraction of a 23 design.
- Because the design contains $2^{3-1} = 4$ treatment combinations, a one-half fraction of the 23 design is often called a 2^{3-1} design.
- Suppose we select the four treatment combinations a, b, c, and abc as our one-half fraction.

■ **TABLE 8.1**

Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−



(*a*) The principal fraction, $I = +ABC$ (*b*) The alternate fraction, $I = -ABC$

■ **FIGURE 8.1** The two one-half fractions of the 2^3 design

- Notice that the 2^{3-1} design is formed by selecting only those treatment combinations that have a plus in the ABC column. Thus, ABC is called the generator of this particular fraction.
- Usually we will refer to a generator such as ABC as a word. Furthermore, the identity column I is also always plus, so we call I=ABC the defining relation for our design. In general, the defining relation for a fractional factorial will always be the set of all columns that are equal to the identity column I.

- the linear combinations of the observations used to estimate the main effects of A, B, and C are

$$[A] = \frac{1}{2}(a - b - c + abc)$$
$$[B] = \frac{1}{2}(-a + b - c + abc)$$
$$[C] = \frac{1}{2}(-a - b + c + abc)$$

- the linear combinations of the observations used to estimate the two-factor interactions are

$$[BC] = \frac{1}{2}(a - b - c + abc)$$
$$[AC] = \frac{1}{2}(-a + b - c + abc)$$
$$[AB] = \frac{1}{2}(-a - b + c + abc)$$

- when we estimate A, B, and C we are really estimating A + BC,B +AC, and C + AB. Two or more effects that have this property are called aliases.
- In our example, A and BC are aliases, B and AC are aliases, and C and AB are aliases.
- The notation for aliases is:

$$[A] \rightarrow A + BC,\ [B] \rightarrow B + AC, \text{ and } [C] \rightarrow C + AB.$$

- Now suppose that we had chosen the other one-half fraction, that is, the treatment combinations in Table 8.1 associated with minus in the ABC column.
  - In practice, it does not matter which fraction is actually used. Both fractions belong to the same family
- This alternate, or complementary, one-half fraction (consisting of the runs (1),ab, ac,and bc) is shown in Figure 8.1b. The defining relation for this design is I=-ABC

The linear combination of the observations, say $[A]'$, $[B]'$, and $[C]'$, from the alternate fraction gives us

$$[A]' \rightarrow A - BC$$
$$[B]' \rightarrow B - AC$$
$$[C]' \rightarrow C - AB$$

Design Resolution

- The preceding 2^{3-1} design is called a resolution III design. In such a design, main effects are aliased with two-factor interactions.
- A design is of resolution R if no p-factor effect is aliased with another effect containing less than R - p factors

We usually employ a Roman numeral subscript to denote design resolution; thus, the one-half fraction of the 2^3 design with the defining relation $I = ABC$ (or $I = -ABC$) is a 2^{3-1}_{III} design.

- Next we discuss the designs of resolution III, IV and V

1. ***Resolution III designs.*** These are designs in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other. The 2^{3-1} design in Table 8.1 is of resolution III (2^{3-1}_{III}).
2. ***Resolution IV designs.*** These are designs in which no main effect is aliased with any other main effect *or* with any two-factor interaction, but two-factor interactions are aliased with each other. A 2^{4-1} design with $I = ABCD$ is a resolution IV design (2^{4-1}_{IV}).
3. ***Resolution V designs.*** These are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions. A 2^{5-1} design with $I = ABCDE$ is a resolution V design (2^{5-1}_{V}).

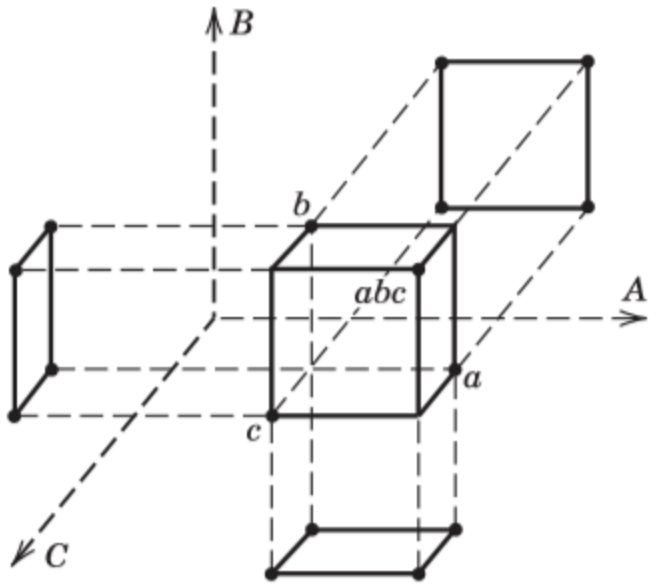
Construction and Analysis of the One-Half Fraction

- A one-half fraction of the 2k design of the highest resolution may be constructed by writing down a basic design consisting of the runs for a full 2k1 factorial and then adding the kth factor by identifying its plus and minus levels with the plus and minus signs of the highest order interaction ABC (K 1).
- Therefore, the fractional factorial is obtained by writing down the full 22 factorial as the basic design and then equating factor C to the AB interaction.
- The alternate fraction would be obtained by equating factor C to the AB interaction.
- The following table will explain the approach

■ **TABLE 8.2**

The Two One-Half Fractions of the 2^3 Design

	Full 2^2 Factorial (Basic Design)		2^{3-1}_{III}, $I = ABC$			2^{3-1}_{III}, $I = -ABC$		
Run	A	B	A	B	$C = AB$	A	B	$C = -AB$
1	−	−	−	−	+	−	−	−
2	+	−	+	−	−	+	−	+
3	−	+	−	+	−	−	+	+
4	+	+	+	+	+	+	+	−



■ **FIGURE 8.2** Projection of a 2^{3-1}_{III} design into three 2^2 designs

Projection of factors into factorials

- Any fractional factorial design of resolution R contains complete factorial designs (possibly replicated factorials) in any subset of R - 1 factors. This is an important and useful concept.
- For example, if an experimenter has several factors of potential interest but believes that only R - 1 of them have important effects, then a fractional factorial design of resolution R is the appropriate choice of design.
- If the experimenter is correct, the fractional factorial design of resolution R will project into a full factorial in the R - 1 significant factors.

Because the maximum possible resolution of a one-half fraction of the 2^k design is $R = k$, every 2^{k-1} design will project into a full factorial in any $(k - 1)$ of the original k factors. Furthermore, a 2^{k-1} design may be projected into two replicates of a full factorial in any subset of $k - 2$ factors, four replicates of a full factorial in any subset of $k - 3$ factors, and so on.

An example

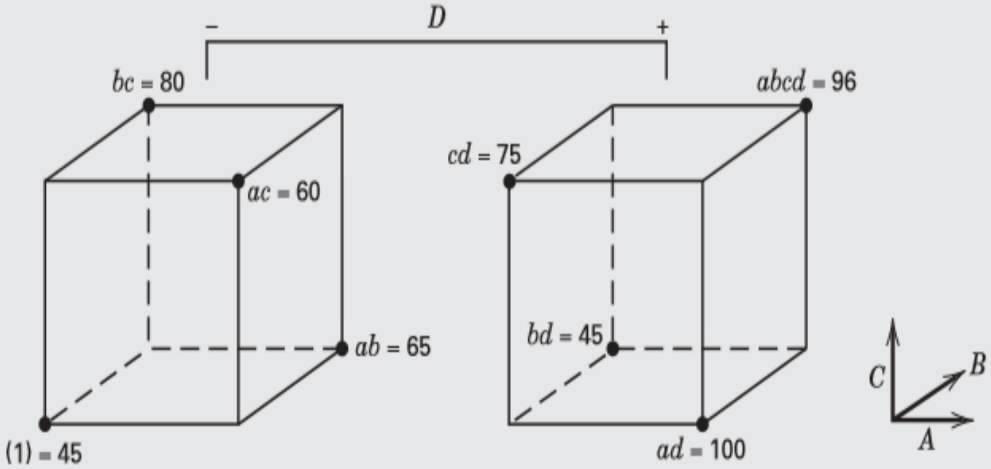
Consider the filtration rate experiment in Example 6.2. The original design, shown in Table 6.10, is a single replicate of the 2^4 design. In that example, we found that the main effects *A*, *C*, and *D* and the interactions *AC* and *AD* were different from zero. We will now return to this experiment and simulate what would have happened if a half-fraction of the 2^4 design had been run instead of the full factorial.

EXAMPLE 6.2 A Single Replicate of the 2^4 Design

A chemical product is produced in a pressure vessel. A factorial experiment is carried out in the pilot plant to study the factors thought to influence the filtration rate of this product. The four factors are temperature (*A*), pressure (*B*), concentration of formaldehyde (*C*), and stirring rate (*D*). Each factor is present at two levels. The design matrix and the response data obtained from a single replicate of the 2^4 experiment are shown in Table 6.10

Run Number	Factor A	B	C	D	Run Label	Filtration Rate (gal/h)
1	−	−	−	−	(1)	45
2	+	−	−	−	*a*	71
3	−	+	−	−	*b*	48
4	+	+	−	−	*ab*	65
5	−	−	+	−	*c*	68
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	−	*abc*	65
9	−	−	−	+	*d*	43
10	+	−	−	+	*ad*	100
11	−	+	−	+	*bd*	45
12	+	+	−	+	*abd*	104
13	−	−	+	+	*cd*	75
14	+	−	+	+	*acd*	86
15	−	+	+	+	*bcd*	70
16	+	+	+	+	*abcd*	96

We will use the 2^{4-1} design with $I = ABCD$, because this choice of generator will result in a design of the highest possible resolution (IV). To construct the design, we first write down the basic design, which is a 2^3 design, as shown in the first three columns of Table 8.3. This basic design has the necessary number of runs (eight) but only three columns (factors). To find the fourth factor levels, solve $I = ABCD$ for D, or $D = ABC$. Thus, the level of D in each run is the product of the plus and minus signs in columns A, B, and C. The process is illustrated in Table 8.3. Because the generator $ABCD$ is positive, this 2_{IV}^{4-1} design is the principal fraction. The design is shown graphically in Figure 8.3.



■ **FIGURE 8.3** The 2_{IV}^{4-1} design for the filtration rate experiment of Example 8.1

■ **TABLE 8.3**

The 2_{IV}^{4-1} Design with the Defining Relation $I = ABCD$

Run	Basic Design			$D = ABC$	Treatment Combination	Filtration Rate
	A	B	C			
1	−	−	−	−	(1)	45
2	+	−	−	+	ad	100
3	−	+	−	+	bd	45
4	+	+	−	−	ab	65
5	−	−	+	+	cd	75
6	+	−	+	−	ac	60
7	−	+	+	−	bc	80
8	+	+	+	+	$abcd$	96

Using the defining relation, we note that each main effect is aliased with a three-factor interaction; that is, $A = A^2BCD = BCD$, $B = AB^2CD = ACD$, $C = ABC^2D = ABD$, and $D = ABCD^2 = ABC$. Furthermore, every two-factor interaction is aliased with another two-factor interaction. These alias relationships are $AB = CD$, $AC = BD$, and $BC = AD$. The four main effects plus the three two-factor interaction alias pairs account for the seven degrees of freedom for the design.

At this point, we would normally randomize the eight runs and perform the experiment. Because we have already run the full 2^4 design, we will simply select the eight observed filtration rates from Example 6.2 that correspond to the runs in the 2^{4-1}_{IV} design. These observations are shown in the last column of Table 8.3 as well as in Figure 8.3.

The estimates of the effects obtained from this 2^{4-1}_{IV} design are shown in Table 8.4. To illustrate the calculations, the linear combination of observations associated with the A effect is

$$[A] = \tfrac{1}{4}(-45 + 100 - 45 + 65 - 75 + 60 - 80 + 96) = 19.00 \rightarrow A + BCD$$

whereas for the AB effect, we would obtain

$$[AB] = \tfrac{1}{4}(45 - 100 - 45 + 65 + 75 - 60 - 80 + 96) = -1.00 \rightarrow AB + CD$$

■ **TABLE 8.4**

Estimates of Effects and Aliases from Example 8.1[a]

Estimate	Alias Structure
$[A] = 19.00$	$[A] \rightarrow A + BCD$
$[B] = 1.50$	$[B] \rightarrow B + ACD$
$[C] = 14.00$	$[C] \rightarrow C + ABD$
$[D] = 16.50$	$[D] \rightarrow D + ABC$
$[AB] = -1.00$	$[AB] \rightarrow AB + CD$
$[AC] = -18.50$	$[AC] \rightarrow AC + BD$
$[AD] = 19.00$	$[AD] \rightarrow AD + BC$

[a] Significant effects are shown in boldface type.

From inspection of the information in Table 8.4, it is not unreasonable to conclude that the main effects A, C, and D are large. The $AB + CD$ alias chain has a small estimate, so the simplest interpretation is that both the AB and CD interactions are negligible (otherwise, both AB and CD are large, but they have nearly identical magnitudes and opposite signs—this is fairly unlikely). Furthermore, if A, C, and D are the important main effects, then it is logical to conclude that the two interaction alias chains $AC + BD$ and $AD + BC$ have large effects because the AC and AD interactions are also significant. In other words, if A, C, and D are significant then the significant interactions are most likely AC and AD. This is an application of **Ockham's razor** (after William of Ockham), a scientific principle that when one is confronted with several different possible interpretations of a phenomena, the simplest interpretation is usually the correct one.

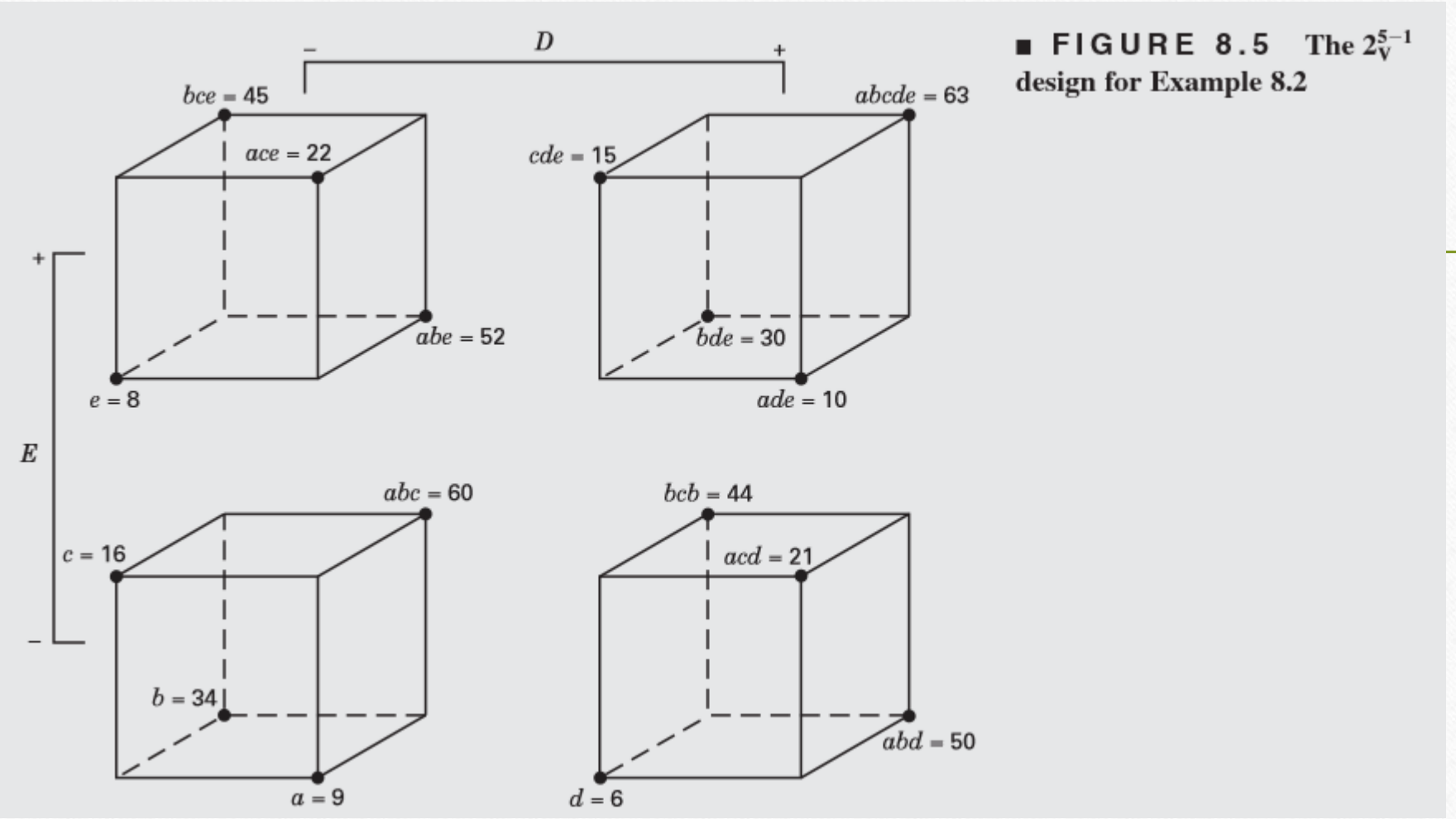
EXAMPLE 8.2 A 2^{5-1} Design Used for Process Improvement

Five factors in a manufacturing process for an integrated circuit were investigated in a 2^{5-1} design with the objective of improving the process yield. The five factors were A = aperture setting (small, large), B = exposure time (20 percent below nominal, 20 percent above nominal), C = develop time (30 and 45 sec), D = mask dimension (small, large), and E = etch time (14.5 and 15.5 min). The construction of the 2^{5-1} design is shown in Table 8.5. Notice that the design was constructed by writing down the basic design having 16 runs (a 2^4 design in A, B, C, and D), selecting $ABCDE$ as the generator, and then setting the levels of the fifth factor $E = ABCD$. Figure 8.5 gives a pictorial representation of the design.

The defining relation for the design is $I = ABCDE$. Consequently, every main effect is aliased with a four-factor interaction (for example, $[A] \rightarrow A + BCDE$), and every two-factor interaction is aliased with a three-factor interaction (e.g., $[AB] \rightarrow AB + CDE$). Thus, the design is of resolution V. We would expect this 2^{5-1} design to provide excellent information concerning the main effects and two-factor interactions.

A 2^{5-1} Design for Example 8.2

Run	Basic Design				$E = ABCD$	Treatment Combination	Yield
	A	*B*	*C*	*D*			
1	−	−	−	−	+	*e*	8
2	+	−	−	−	−	*a*	9
3	−	+	−	−	−	*b*	34
4	+	+	−	−	+	*abe*	52
5	−	−	+	−	−	*c*	16
6	+	−	+	−	+	*ace*	22
7	−	+	+	−	+	*bce*	45
8	+	+	+	−	−	*abc*	60
9	−	−	−	+	−	*d*	6
10	+	−	−	+	+	*ade*	10
11	−	+	−	+	+	*bde*	30
12	+	+	−	+	−	*abd*	50
13	−	−	+	+	+	*cde*	15
14	+	−	+	+	−	*acd*	21
15	−	+	+	+	−	*bcd*	44
16	+	+	+	+	+	*abcde*	63



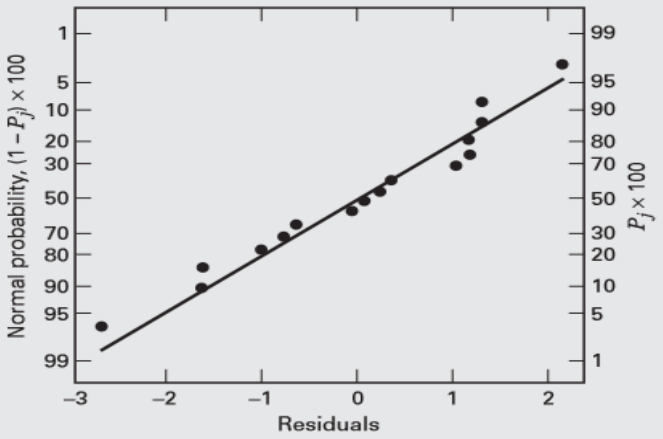
■ **FIGURE 8.5** The 2_V^{5-1} design for Example 8.2

Table 8.7 summarizes the analysis of variance for this experiment. The model sum of squares is $SS_{\text{Model}} = SS_A + SS_B + SS_C + SS_{AB} = 5747.25$, and this accounts for over 99 percent of the total variability in yield. Figure 8.7 presents a normal probability plot of the residuals, and Figure 8.8 is a plot of the residuals versus the predicted values. Both plots are satisfactory.

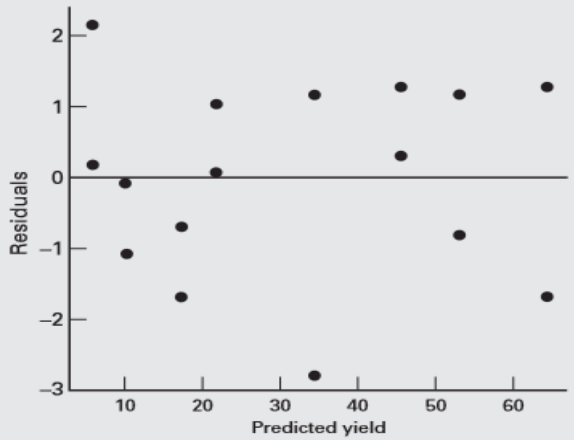
■ **TABLE 8.7**

Analysis of Variance for Example 8.2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
A (Aperture)	495.0625	1	495.0625	193.20	<0.0001
B (Exposure time)	4590.0625	1	4590.0625	1791.24	<0.0001
C (Develop time)	473.0625	1	473.0625	184.61	<0.0001
AB	189.0625	1	189.0625	73.78	<0.0001
Error	28.1875	11	2.5625		
Total	5775.4375	15			



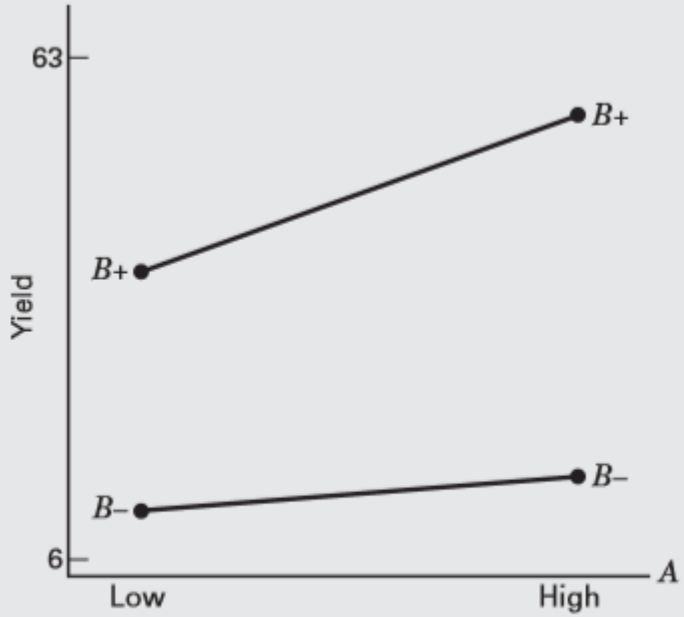
■ **FIGURE 8.7** Normal probability plot of the residuals for Example 8.2



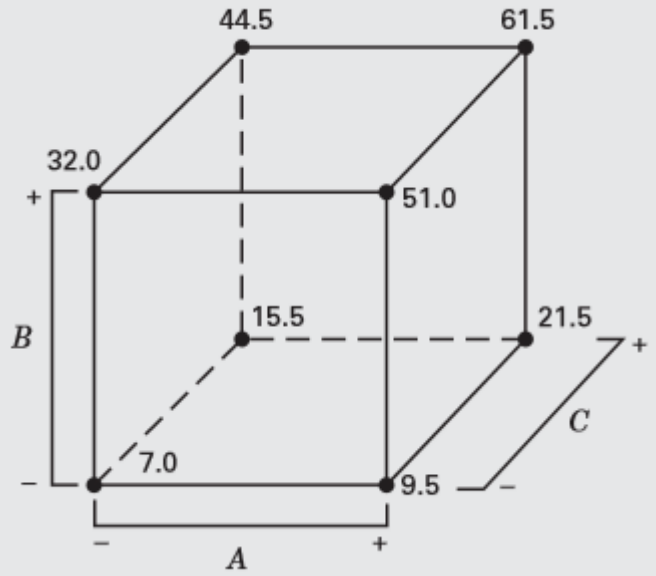
■ **FIGURE 8.8** Plot of residuals versus predicted yield for Example 8.2

The three factors A, B, and C have large positive effects. The AB or aperture–exposure time interaction is plotted in Figure 8.9. This plot confirms that the yields are higher when both A and B are at the high level.

The 2^{5-1} design will collapse into two replicates of a 2^3 design in any three of the original five factors. (Looking at Figure 8.5 will help you visualize this.) Figure 8.10 is a cube plot in the factors A, B, and C with the average yields superimposed on the eight corners. It is clear from inspection of the cube plot that highest yields are achieved with A, B, and C all at the high level. Factors D and E have little effect on average process yield and may be set to values that optimize other objectives (such as cost).



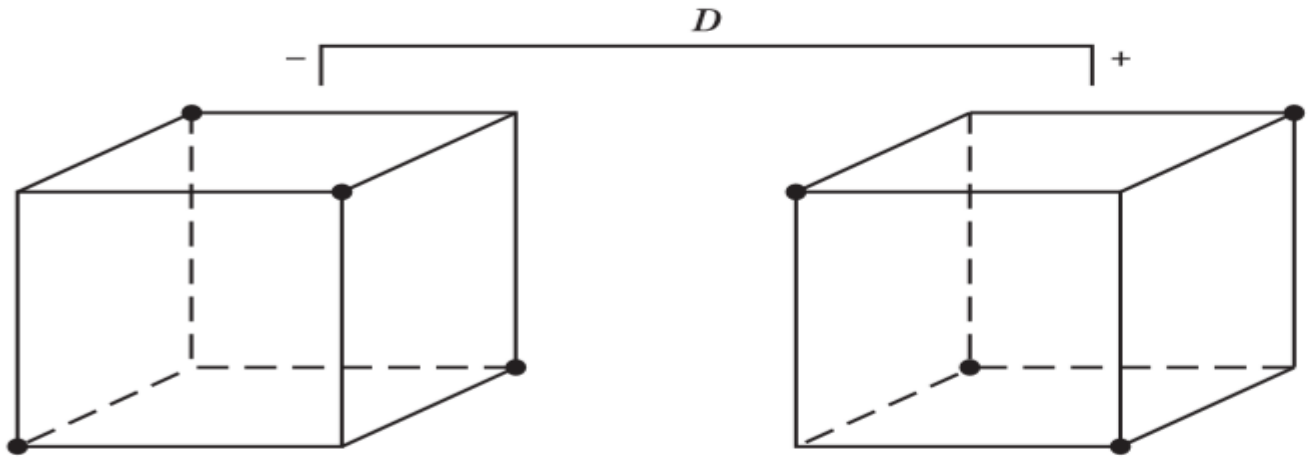
■ **FIGURE 8.9** **Aperture–exposure time interaction for Example 8.2**



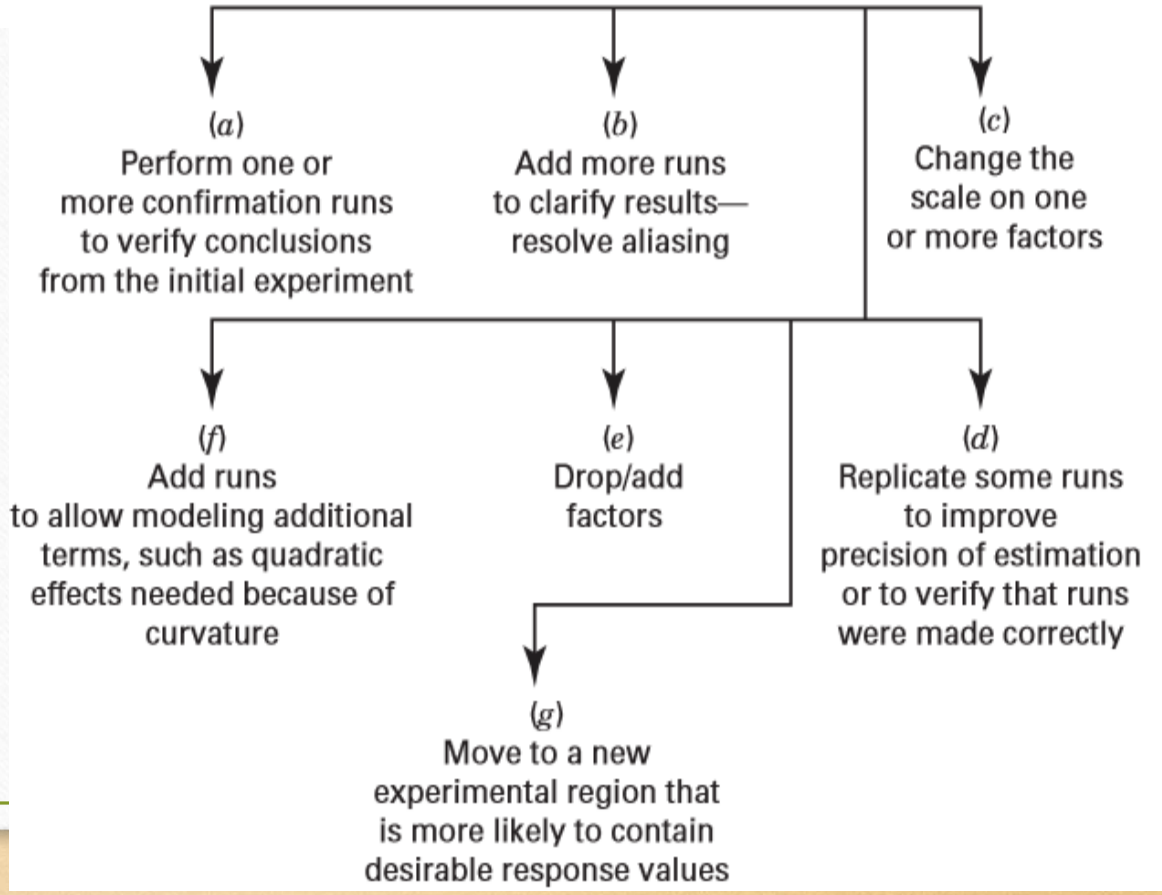
■ **FIGURE 8.10** **Projection of the 2_V^{5-1} design in Example 8.2 into two replicates of a 2^3 design in the factors A, B, and C**

Sequences of Fractional Factorials

- Using fractional factorial designs often leads to great economy and efficiency in experimentation, particularly if the runs can be made sequentially.
- For example, suppose that we are investigating k 4 factors (24 16 runs). It is almost always preferable to run a 2^{4-1}_{IV} fractional design (eight runs), analyze the results, and then decide on the best set of runs to perform next.
- If it is necessary to resolve ambiguities, we can always run the alternate fraction and complete the 24 design.
-



■ **FIGURE 8.11 Possibilities for follow-up experimentation after an initial fractional factorial experiment**



The One-Quarter Fraction of the 2k Design

- For a moderately large number of factors, smaller fractions of the 2^k design are frequently useful.
- Consider a one-quarter fraction of the 2^k design. This design contains 2^{k-2} runs and is usually called a 2^{k-2} fractional factorial
- The 2^{k-2} design may be constructed by first writing down a basic design consisting of the runs associated with a full factorial in k - 2 factors and then associating the two additional columns with appropriately chosen interactions involving the first k - 2 factors.
- Thus, a one-quarter fraction of the 2^k design has two generators. If P and Q represent the generators chosen, then I=P and I =Q are called the generating relations for the design. The signs of P and Q (either + or -) determine which one of the one-quarter fractions is produced.
- All four fractions associated with the choice of generators $\pm$ P and $\pm$ Q are members of the same family. The fraction for which both P and Q are positive is the principal fraction.

- The complete defining relation for the design consists of all the columns that are equal to the identity column I.
- These will consist of P, Q, and their generalized interaction PQ; that is, the defining relation is I = P = Q = PQ.
- We call the elements P, Q, and PQ in the defining relation **words**. The aliases of any effect are produced by the multiplication of the column for that effect by each word in the defining relation.

EXAMPLE 8.4

Parts manufactured in an injection molding process are showing excessive shrinkage. This is causing problems in assembly operations downstream from the injection molding area. A quality improvement team has decided to use a designed experiment to study the injection molding process so that shrinkage can be reduced. The team decides to investigate six factors—mold temperature (A), screw speed (B), holding time (C), cycle time (D), gate size (E), and holding pressure (F)—each at two levels, with the objective of learning how each factor affects shrinkage and also, something about how the factors interact.

The team decides to use the 16-run two-level fractional factorial design in Table 8.9. The design is shown again in Table 8.10, along with the observed shrinkage ($\times 10$) for the test part produced at each of the 16 runs in the design. Table 8.11 shows the effect estimates, sums of squares, and the regression coefficients for this experiment.

A normal probability plot of the effect estimates from this experiment is shown in Figure 8.12. The only large effects are A (mold temperature), B (screw speed), and the AB interaction. In light of the alias relationships in Table 8.8, it seems reasonable to adopt these conclusions tentatively. The plot of the AB interaction in Figure 8.13 shows that the process is very insensitive to temperature if the screw speed is at the low level but very sensitive to temperature if the screw speed is at the high level. With the screw speed at the low level, the process should produce an average shrinkage of around 10 percent regardless of the temperature level chosen.

Based on this initial analysis, the team decides to set both the mold temperature and the screw speed at the low level. This set of conditions will reduce the *mean* shrinkage of parts to around 10 percent. However, the variability in shrinkage from part to part is still a potential problem. In effect, the mean shrinkage can be adequately reduced by the above modifications; however, the part-to-part variability in shrinkage over a production run could still cause problems in assembly. One way to address this issue is to see if any of the process factors affect the *variability* in parts shrinkage.

A 2_{IV}^{6-2} Design for the Injection Molding Experiment in Example 8.4

Run	Basic Design				$E = ABC$	$F = BCD$	Observed Shrinkage (× 10)	
	A	*B*	*C*	*D*				
1	−	−	−	−	−	−	6	(1)
2	+	−	−	−	+	−	10	*ae*
3	−	+	−	−	+	+	32	*bef*
4	+	+	−	−	−	+	60	*abf*
5	−	−	+	−	+	+	4	*cef*
6	+	−	+	−	−	+	15	*acf*
7	−	+	+	−	−	−	26	*bc*
8	+	+	+	−	+	−	60	*abce*
9	−	−	−	+	−	+	8	*df*
10	+	−	−	+	+	+	12	*adef*
11	−	+	−	+	+	−	34	*bde*
12	+	+	−	+	−	−	60	*abd*
13	−	−	+	+	+	−	16	*cde*
14	+	−	+	+	−	−	5	*acd*
15	−	+	+	+	−	+	37	*bcdf*
16	+	+	+	+	+	+	52	*abcdef*

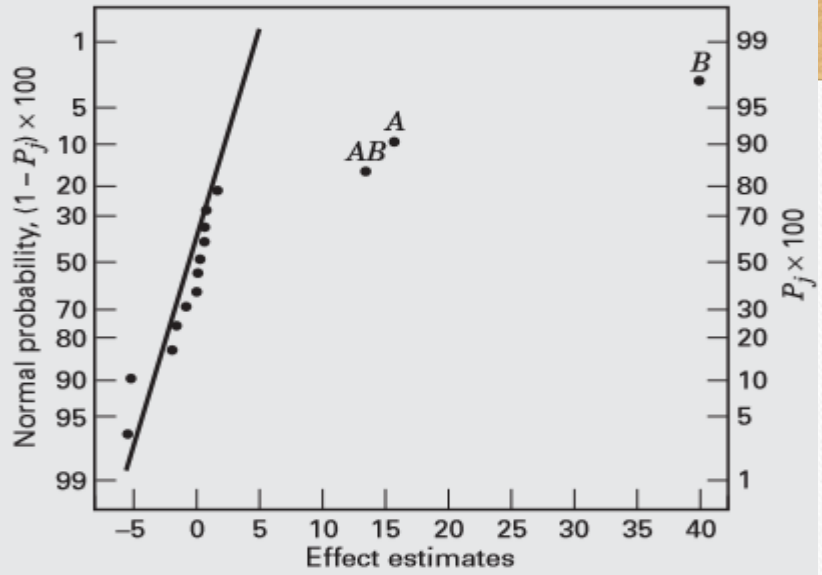
■ **TABLE 8.11**

Effects, Sums of Squares, and Regression Coefficients for Example 8.4

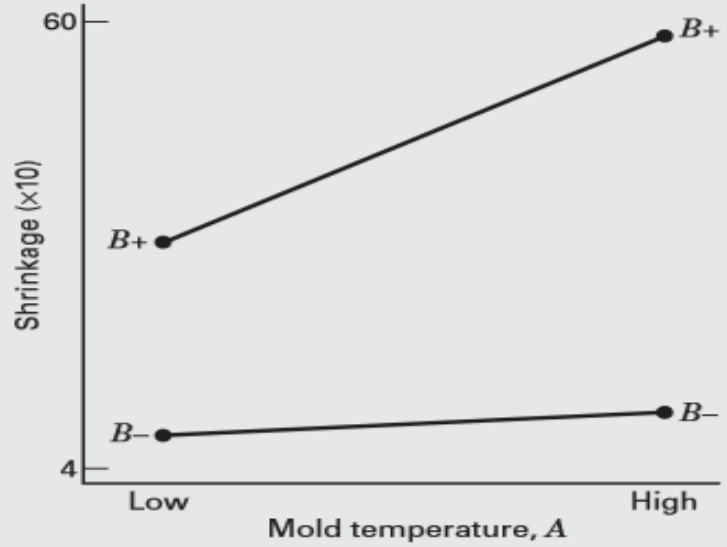
Variable	Name	−1 Level	+1 Level
A	Mold temperature	−1.000	1.000
B	Screw speed	−1.000	1.000
C	Holding time	−1.000	1.000
D	Cycle time	−1.000	1.000
E	Gate size	−1.000	1.000
F	Hold pressure	−1.000	1.000

Variable[a]	Regression Coefficient	Estimated Effect	Sum of Squares
Overall Average	27.3125		
A	6.9375	13.8750	770.062
B	17.8125	35.6250	5076.562
C	−0.4375	−0.8750	3.063
D	0.6875	1.3750	7.563
E	0.1875	0.3750	0.563
F	0.1875	0.3750	0.563
AB + CE	5.9375	11.8750	564.063
AC + BE	−0.8125	−1.6250	10.562
AD + EF	−2.6875	−5.3750	115.562
AE + BC + DF	−0.9375	−1.8750	14.063
AF + DE	0.3125	0.6250	1.563
BD + CF	−0.0625	−0.1250	0.063
BF + CD	−0.0625	−0.1250	0.063
ABD	0.0625	0.1250	0.063
ABF	−2.4375	−4.8750	95.063

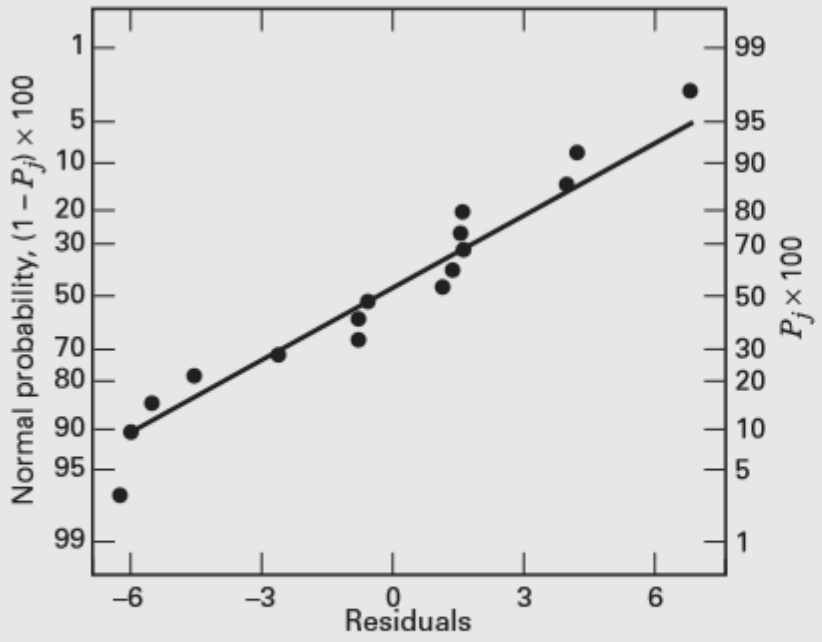
[a]Only main effects and two-factor interactions.



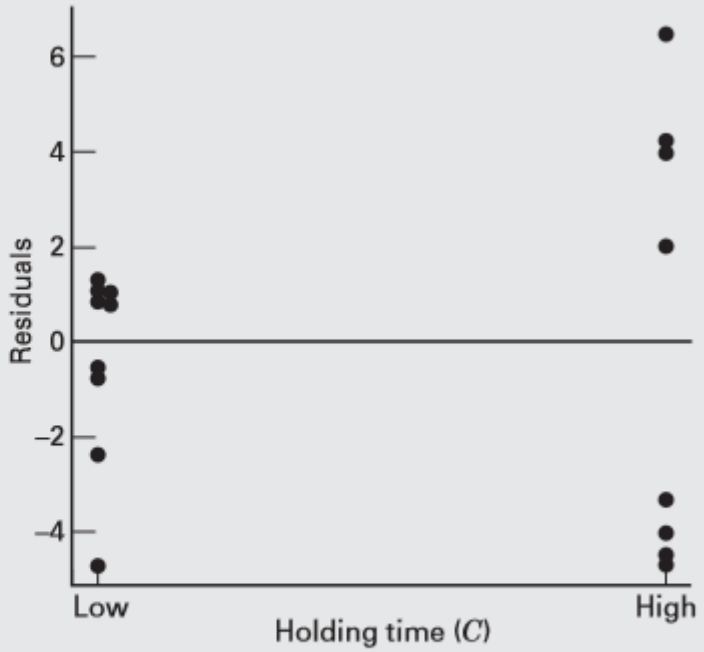
■ **FIGURE 8.12** **Normal probability plot of effects for Example 8.4**



■ **FIGURE 8.13** **Plot of *AB* (mold temperature–screw speed) interaction for Example 8.4**



■ **FIGURE 8.14** **Normal probability plot of residuals for Example 8.4**



■ **FIGURE 8.15** **Residuals versus holding time (*C*) for Example 8.4**

Figure 8.14 presents the normal probability plot of the residuals. This plot appears satisfactory. The plots of residuals versus each factor were then constructed. One of these plots, that for residuals versus factor C (holding time), is shown in Figure 8.15. The plot reveals that there is much less scatter in the residuals at the low holding time than at the high holding time. These residuals were obtained in the usual way from a model for predicted shrinkage:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_{12} x_1 x_2$$
$$= 27.3125 + 6.9375 x_1 + 17.8125 x_2 + 5.9375 x_1 x_2$$

where x_1, x_2, and $x_1 x_2$ are coded variables that correspond to the factors A and B and the AB interaction. The residuals are then

$$e = y - \hat{y}$$

The regression model used to produce the residuals essentially removes the **location effects** of A, B, and AB from the data; the residuals therefore contain information about unexplained variability. Figure 8.15 indicates that there is a *pattern* in the variability and that the variability in the shrinkage of parts may be smaller when the holding time is at the low level. (Please recall that we observed in Chapter 6 that residuals only convey information about dispersion effects when the location or mean model is correct.)

This is further amplified by the analysis of residuals shown in Table 8.12. In this table, the residuals are arranged at the low (−) and high (+) levels of each factor, and the standard deviations of the residuals at the low and high levels of each factor have been calculated. Note that the standard deviation of the residuals with C at the low level $[S(C^-) = 1.63]$ is considerably smaller than the standard deviation of the residuals with C at the high level $[S(C^+) = 5.70]$.

The bottom line of Table 8.12 presents the statistic

$$F_i^* = \ln \frac{S^2(i^+)}{S^2(i^-)}$$

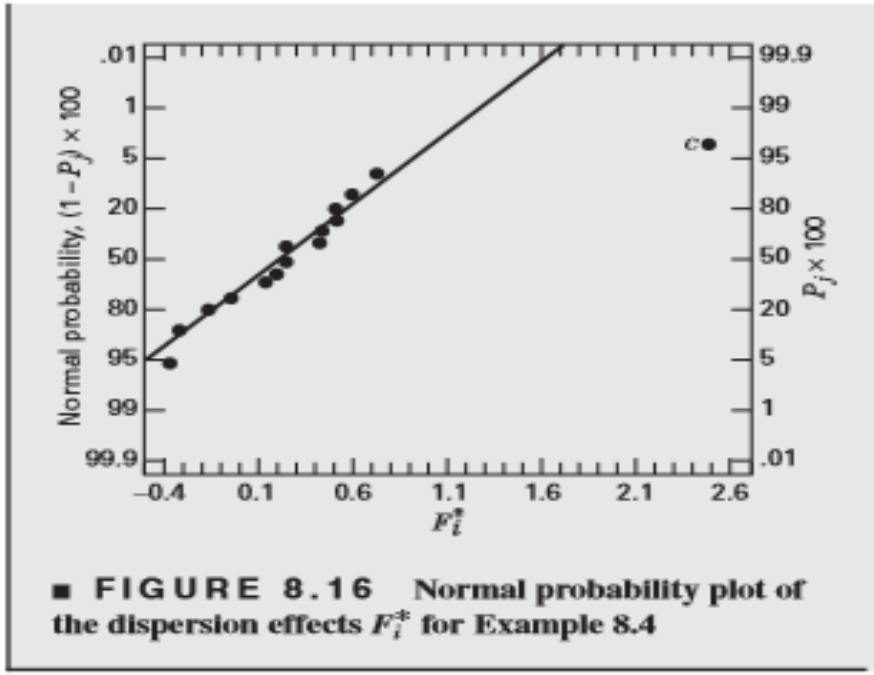
Recall that if the variances of the residuals at the high (+) and low (−) levels of factor i are equal, then this ratio is approximately normally distributed with mean zero, and it can be used to judge the difference in the response variability at the two levels of factor i. Because the ratio F_C^* is relatively large, we would conclude that the apparent **dispersion** or **variability effect** observed in Figure 8.15 is real. Thus, setting the holding time at its low level would contribute to reducing the variability in shrinkage from part to part during a production run. Figure 8.16 presents a normal probability plot of the F_i^* values in Table 8.12; this also indicates that factor C has a large dispersion effect.

Figure 8.17 shows the data from this experiment projected onto a cube in the factors A, B, and C. The average observed shrinkage and the range of observed shrinkage are shown at each corner of the cube. From inspection of this figure, we see that running the process with the screw speed (B) at the low level is the key to reducing average parts shrinkage. If B is low, virtually any combination of temperature (A) and holding time (C) will result in low values of average parts shrinkage. However, from examining the ranges of the shrinkage values at each corner of the cube, it is immediately clear that setting the holding time (C) at the low level is the only reasonable choice if we wish to keep the part-to-part variability in shrinkage low during a production run.

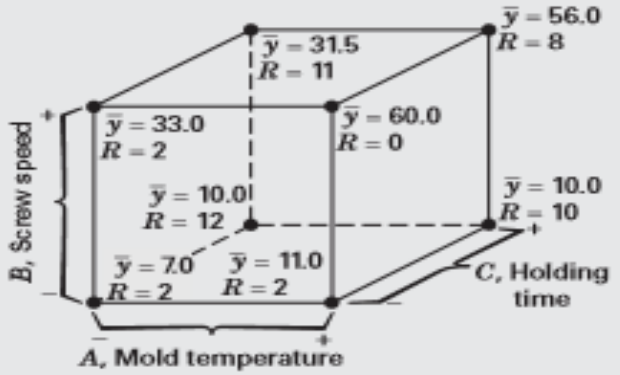
■ TABLE 8.12

Calculation of Dispersion Effects for Example 8.4

Run	*A*	*B*	*AB = CE*	*C*	*AC = BE*	*AE = BC = DF*	*E*	*D*	*AD = EF*	*BD = CE*	*ABD*	*BF = CD*	*ACD*	*F*	*AF = DE*	Residual
1	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+	−2.50
2	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−	−0.50
3	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−	−0.25
4	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+	2.00
5	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−	−4.50
6	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+	4.50
7	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+	−6.25
8	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	2.00
9	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−	−0.50
10	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+	1.50
11	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+	1.75
12	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−	2.00
13	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+	7.50
14	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	−5.50
15	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	4.75
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−6.00
$S(i^+)$	3.80	4.01	4.33	5.70	3.68	3.85	4.17	4.64	3.39	4.01	4.72	4.71	3.50	3.88	4.87	
$S(i^-)$	4.60	4.41	4.10	1.63	4.53	4.33	4.25	3.59	2.75	4.41	3.64	3.65	3.12	4.52	3.40	
F_i^*	−0.38	−0.19	0.11	2.50	−0.42	−0.23	−0.04	0.51	0.42	−0.19	0.52	0.51	0.23	−0.31	0.72	



■ **FIGURE 8.16** Normal probability plot of the dispersion effects F_i^* for Example 8.4



■ **FIGURE 8.17** Average shrinkage and range of shrinkage in factors A, B, and C for Example 8.4

The General 2^{k-p} Fractional Factorial Design

- A 2^k fractional factorial design containing 2^{k-p} runs is called a $1/2^p$ fraction of the 2^k design or, more simply, a 2^{k-p} fractional factorial design.
- These designs require the selection of p independent generators. The defining relation for the design consists of the p generators initially chosen and their 2^p - p - 1 generalized interactions.
- The alias structure may be found by multiplying each effect column by the defining relation.
- Care should be exercised in choosing the generators so that effects of potential interest are not aliased with each other. Each effect has 2^p - 1 aliases.

- It is important to select the p generators for a 2^{k-p} fractional factorial design in such a way that we obtain the best possible alias relationships.
- A reasonable criterion is to select the generators such that the resulting 2^{k-p} design has the **highest possible resolution**.
- Lets look at an example

■ TABLE 8.9

Construction of the 2^{6-2}_{IV} Design with the Generators $I = ABCE$ and $I = BCDF$

	Basic Design					
Run	*A*	*B*	*C*	*D*	$E = ABC$	$F = BCD$
1	−	−	−	−	−	−
2	+	−	−	−	+	−
3	−	+	−	−	+	+
4	+	+	−	−	−	+
5	−	−	+	−	+	+
6	+	−	+	−	−	+
7	−	+	+	−	−	−
8	+	+	+	−	+	−
9	−	−	−	+	−	+
10	+	−	−	+	+	+
11	−	+	−	+	+	−
12	+	+	−	+	−	−
13	−	−	+	+	+	−
14	+	−	+	+	−	−
15	−	+	+	+	−	+
16	+	+	+	+	+	+

- Here we used the generators E = ABC and F = BCD, thereby producing a design of resolution IV.
- This is the maximum resolution design. If we had selected E = ABC and F = ABCD, the complete defining relation would have been I = ABCE = ABCDF = DEF, and the design would be of resolution III.
- Clearly, this is an inferior choice because it needlessly sacrifices information about interactions.

- Sometimes resolution alone is insufficient to distinguish between designs. For example, consider the three designs in Table 8.13 (given in next slide).
- All of these designs are of resolution IV, but they have rather different alias structures (we have assumed that three-factor and higher interactions are negligible) with respect to the two-factor interactions.
- Clearly, design A has more extensive aliasing and design C the least, so design C would be the best choice.
- The three word lengths in design A are all 4; that is, the word length pattern is {4, 4, 4}. For design B it is {4, 4, 6}, and for design C it is {4, 5, 5}. Notice that the defining relation for design C has only one four-letter word, whereas the other designs have two or three.

■ **TABLE 8.13**

Three Choices of Generators for the 2_{IV}^{7-2} Design

Design *A* Generators: $F = ABC, G = BCD$ $I = ABCF = BCDG = ADFG$	**Design *B* Generators:** $F = ABC, G = ADE$ $I = ABCF = ADEG = BCDEFG$	**Design *C* Generators:** $F = ABCD, G = ABDE$ $I = ABCDF = ABDEG = CEFG$
Aliases (two-factor interactions)	Aliases (two-factor interactions)	Aliases (two-factor interactions)
$AB = CF$	$AB = CF$	$CE = FG$
$AC = BF$	$AC = BF$	$CF = EG$
$AD = FG$	$AD = EG$	$CG = EF$
$AG = DF$	$AE = DG$	
$BD = CG$	$AF = BC$	
$BG = CD$	$AG = DE$	
$AF = BC = DG$		

- Thus, design C minimizes the number of words in the defining relation that are of minimum length. We call such a design a minimum aberration design.
- Minimizing aberration in a design of resolution R ensures that the design has the minimum number of main effects aliased with interactions of order R - 1, the minimum number of two-factor interactions aliased with interactions of order R - 2, and so forth.

- The following table (in the next slide) presents a selection of 2^{k-p} fractional factorial designs for k <= 15 factors and up to n <= 128 runs. The suggested generators in this table will result in a design of the highest possible resolution. These are also the minimum aberration designs.

Number of Factors, k	Fraction	Number of Runs	Design Generators
3	2^{3-1}_{III}	4	$C = \pm AB$
4	2^{4-1}_{IV}	8	$D = \pm ABC$
5	2^{5-1}_{V}	16	$E = \pm ABCD$
	2^{5-2}_{III}	8	$D = \pm AB$
			$E = \pm AC$
6	2^{6-1}_{VI}	32	$F = \pm ABCDE$
	2^{6-2}_{IV}	16	$E = \pm ABC$
			$F = \pm BCD$
	2^{6-3}_{III}	8	$D = \pm AB$
			$E = \pm AC$
			$F = \pm BC$
7	2^{7-1}_{VII}	64	$G = \pm ABCDEF$
	2^{7-2}_{IV}	32	$F = \pm ABCD$
			$G = \pm ABDE$
	2^{7-3}_{IV}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
	2^{7-4}_{III}	8	$D = \pm AB$
			$E = \pm AC$
			$F = \pm BC$
			$G = \pm ABC$
8	2^{8-2}_{V}	64	$G = \pm ABCD$
			$H = \pm ABEF$
	2^{8-3}_{IV}	32	$F = \pm ABC$
			$G = \pm ABD$
			$H = \pm BCDE$
	2^{8-4}_{IV}	16	$E = \pm BCD$
			$F = \pm ACD$
			$G = \pm ABC$
			$H = \pm ABD$
9	2^{9-2}_{VI}	128	$H = \pm ACDFG$
			$J = \pm BCEFG$
	2^{9-3}_{IV}	64	$G = \pm ABCD$
			$H = \pm ACEF$
			$J = \pm CDEF$
	2^{9-4}_{IV}	32	$F = \pm BCDE$
			$G = \pm ACDE$
			$H = \pm ABDE$
			$J = \pm ABCE$
	2^{9-5}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
10	2^{10-3}_{V}	128	$H = \pm ABCG$
			$J = \pm BCDE$
			$K = \pm ACDF$
	2^{10-4}_{IV}	64	$G = \pm BCDF$
			$H = \pm ACDF$
			$J = \pm ABDE$
			$K = \pm ABCE$
	2^{10-5}_{IV}	32	$F = \pm ABCD$
			$G = \pm ABCE$
			$H = \pm ABDE$
			$J = \pm ACDE$
			$K = \pm BCDE$
	2^{10-6}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
			$K = \pm AB$
11	2^{11-5}_{IV}	64	$G = \pm CDE$
			$H = \pm ABCD$
			$J = \pm ABF$
			$K = \pm BDEF$
			$L = \pm ADEF$
	2^{11-6}_{IV}	32	$F = \pm ABC$
			$G = \pm BCD$
			$H = \pm CDE$
			$J = \pm ACD$
			$K = \pm ADE$
			$L = \pm BDE$
	2^{11-7}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
			$K = \pm AB$

EXAMPLE 8.5

To illustrate the use of Table 8.14, suppose that we have seven factors and that we are interested in estimating the seven main effects and getting some insight regarding the two-factor interactions. We are willing to assume that three-factor and higher interactions are negligible. This information suggests that a resolution IV design would be appropriate.

Table 8.14 shows that there are two resolution IV fractions available: the 2_{IV}^{7-2} with 32 runs and the 2_{IV}^{7-3} with 16 runs. Appendix Table X contains the complete alias relationships for these two designs. The aliases for the 2_{IV}^{7-3} 16-run design are in Appendix Table X(i). Notice that all seven main effects are aliased with three-factor interactions. The two-factor interactions are all aliased in groups of three. Therefore, this design will satisfy our objectives; that is, it will allow the estimation of the main effects, and it will give some insight regarding two-factor interactions. It is not necessary to run the 2_{IV}^{7-2} design, which would require 32 runs. Appendix Table X(j) shows that this design would allow the estimation of all seven main effects and that 15 of the 21 two-factor interactions could also be uniquely estimated. (Recall that three-factor and higher interactions are negligible.) This is probably more information about interactions than is necessary. The complete 2_{IV}^{7-3} design is shown in Table 8.15. Notice that it was constructed by starting with the 16-run 2^4 design in A, B, C, and D as the basic design and then adding the three columns $E = ABC$, $F = BCD$, and $G = ACD$. The generators are $I = ABCE$, $I = BCDF$, and $I = ACDG$ (Table 8.14). The complete defining relation is $I = ABCE = BCDF = ADEF = ACDG = BDEG = CEFG = ABFG$.

■ TABLE 8.15

A 2_{IV}^{7-3} Fractional Factorial Design

	Basic Design						
Run	*A*	*B*	*C*	*D*	*E* = *ABC*	*F* = *BCD*	*G* = *ACD*
1	−	−	−	−	−	−	−
2	+	−	−	−	+	−	+
3	−	+	−	−	+	+	−
4	+	+	−	−	−	+	+
5	−	−	+	−	+	+	+
6	+	−	+	−	−	+	−
7	−	+	+	−	−	−	+
8	+	+	+	−	+	−	−
9	−	−	−	+	−	+	+
10	+	−	−	+	+	+	−
11	−	+	−	+	+	−	+
12	+	+	−	+	−	−	−
13	−	−	+	+	+	−	−
14	+	−	+	+	−	−	+
15	−	+	+	+	−	+	−
16	+	+	+	+	+	+	+

Blocking Fractional Factorials

- Occasionally, a fractional factorial design requires so many runs that all of them cannot be made under homogeneous conditions. In these situations, fractional factorials may be confounded in blocks.
- consider the fractional factorial design with the defining relation I = ABCE = BCDF =ADEF
- This fractional design contains 16 treatment combinations.
- Suppose we wish to run the design in two blocks of eight treatment combinations each. In selecting an interaction to confound with blocks, we note from examining the alias structure in Appendix Table X(f)(part extracted in the next slide) that there are two alias sets involving only three-factor interactions.
- The table suggests selecting ABD(and its aliases) to be confounded with blocks.

(f) 2^{6-2}; 1/4 fraction of
6 factors in 16 runs

Resolution IV

Design Generators

$E = ABC \quad F = BCD$

Defining relation: $I = ABCE = BCDF = ADEF$

Aliases

$A = BCE = DEF$	$AB = CE$
$B = ACE = CDF$	$AC = BE$
$C = ABE = BDF$	$AD = EF$
$D = BCF = AEF$	$AE = BC = DF$
$E = ABC = ADF$	$AF = DE$
$F = BCD = ADE$	$BD = CF$
$ABD = CDE = ACF = BEF$	$BF = CD$
$ACD = BDE = ABF = CEF$	

2 blocks of 8: $ABD = CDE = ACF = BEF$

Block 1	Block 2
(1)	*ae*
abf	*acf*
cef	*bef*
abce	*bc*
adef	*df*
bde	*abd*
acd	*cde*
bcdf	*abcdef*

■ **FIGURE 8.18** **The 2^{6-2}_{IV} design in two blocks with *ABD* confounded**

Alias Structures in Fractional Factorials and Other Designs

- The method uses the polynomial or regression model representation of the model, say $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}$
- where $\mathbf{y}$ is an $n \times 1$ vector of the responses, $\mathbf{X}_1$ is an $n \times p_1$ matrix containing the design matrix expanded to the form of the model that the experimenter is fitting, $\boldsymbol{\beta}_1$ is a $p_1 \times 1$ vector of the model parameters, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of errors. The least squares estimate of $\boldsymbol{\beta}_1$ is

$$\hat{\boldsymbol{\beta}}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y}$$

Suppose that the **true** model is

$$\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$$

where $\mathbf{X}_2$ is an $n \times p_2$ matrix containing additional variables that are not in the fitted model and $\boldsymbol{\beta}_2$ is a $p_2 \times 1$ vector of the parameters associated with these variables. It can be shown that

$$\begin{aligned} E(\hat{\boldsymbol{\beta}}_1) &= \boldsymbol{\beta}_1 + (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2 \\ &= \boldsymbol{\beta}_1 + \mathbf{A}\boldsymbol{\beta}_2 \end{aligned} \tag{8.1}$$

The matrix $\mathbf{A} = (\mathbf{X}_1'\,\mathbf{X}_1)^{-1}\mathbf{X}_1'\,\mathbf{X}_2$ is called the **alias matrix**. The elements of this matrix operating on $\boldsymbol{\beta}_2$ identify the alias relationships for the parameters in the vector $\boldsymbol{\beta}_1$.

Resolution III designs

- It is possible to construct resolution III designs for investigating up to k = N - 1 factors in only N runs, where N is a multiple of 4.
- These designs are frequently useful in industrial experimentation.
- Designs in which N is a power of 2 can be constructed by the methods presented earlier
- Of particular importance are designs requiring 4 runs for up to 3 factors, 8 runs for up to 7 factors, and 16 runs for up to 15 factors. If k = N - 1, the fractional factorial design is said to be saturated.

- A very useful saturated fractional factorial is a design for studying seven factors in eight runs, that is, the 2_{III}^{7-4} design
- This design is a one-sixteenth fraction of the 2^7. It may be constructed by first writing down as the basic design the plus and minus levels for a full 2^3 design in A, B, and C and then associating the levels of four additional factors with the interactions of the original three as follows: D = AB, E = AC, F = BC, and G = ABC. Thus, the generators for this design are I = ABD, I = ACE, I = BCF, and I = ABCG.

■ **TABLE 8.19**

The 2_{III}^{7-4} Design with the Generators $I = ABD$, $I = ACE$, $I = BCF$, and $I = ABCG$

	Basic Design							
Run	***A***	***B***	***C***	***D = AB***	***E = AC***	***F = BC***	***G = ABC***	
1	−	−	−	+	+	+	−	*def*
2	+	−	−	−	−	+	+	*afg*
3	−	+	−	−	+	−	+	*beg*
4	+	+	−	+	−	−	−	*abd*
5	−	−	+	+	−	−	+	*cdg*
6	+	−	+	−	+	−	−	*ace*
7	−	+	+	−	−	+	−	*bcf*
8	+	+	+	+	+	+	+	*abcdefg*

The complete defining relation for this design is obtained by multiplying the four generators *ABD, ACE, BCF*, and *ABCG* together two at a time, three at a time, and four at a time, yielding

$$I = ABD = ACE = BCF = ABCG = BCDE = ACDF = CDG$$
$$= ABEF = BEG = AFG = DEF = ADEG = CEFG = BDFG = ABCDEFG$$

To find the aliases of any effect, simply multiply the effect by each word in the defining relation. For example, the aliases of *B* are

$$B = AD = ABCE = CF = ACG = CDE = ABCDF = BCDG = AEF = EG$$
$$= ABFG = BDEF = ABDEG = BCEFG = DFG = ACDEFG$$

- If we assume that three-factor and higher interactions are negligible, then :

$$[A] \rightarrow A + BD + CE + FG$$

$$[B] \rightarrow B + AD + CF + EG$$

$$[C] \rightarrow C + AE + BF + DG$$

$$[D] \rightarrow D + AB + CG + EF$$

$$[E] \rightarrow E + AC + BG + DF$$

$$[F] \rightarrow F + BC + AG + DE$$

$$[G] \rightarrow G + CD + BE + AF$$

The saturated 2_{III}^{7-4} design in Table 8.19 can be used to obtain resolution III designs for studying fewer than seven factors in eight runs. For example, to generate a design for six factors in eight runs, simply drop any one column in Table 8.19, for example, column *G*. This produces the design shown in Table 8.20.

A 2_{III}^{6-3} Design with the Generators $I = ABD$, $I = ACE$, and $I = BCF$

	Basic Design						
Run	***A***	***B***	***C***	***D = AB***	***E = AC***	***F = BC***	
1	−	−	−	+	+	+	*def*
2	+	−	−	−	−	+	*af*
3	−	+	−	−	+	−	*be*
4	+	+	−	+	−	−	*abd*
5	−	−	+	+	−	−	*cd*
6	+	−	+	−	+	−	*ace*
7	−	+	+	−	−	+	*bcf*
8	+	+	+	+	+	+	*abcdef*

Fold Over of Resolution III Fractions to Separate Aliased Effects

- By combining fractional factorial designs in which certain signs are switched, we can systematically isolate effects of potential interest.
- This type of sequential experiment is called a fold over of the original design.
- The alias structure for any fraction with the signs for one or more factors reversed is obtained by making changes of sign on the appropriate factors in the alias structure of the original fraction.

Consider the 2_{III}^{7-4} design in Table 8.19. Suppose that along with this principal fraction a second fractional design with the signs reversed in the column for factor D is also run. That is, the column for D in the second fraction is

$$-++--++-$$

The effects that may be estimated from the first fraction are shown in Equation 8.2, and from the second fraction we obtain

$$[A]' \rightarrow A - BD + CE + FG$$

$$[B]' \rightarrow B - AD + CF + EG$$

$$[C]' \rightarrow C + AE + BF - DG$$

$$[D]' \rightarrow D - AB - CG - EF$$

$$[-D]' \rightarrow -D + AB + CG + EF$$

$$[E]' \rightarrow E + AC + BG - DF$$

$$[F]' \rightarrow F + BC + AG - DE$$

$$[G]' \rightarrow G - CD + BE + AF$$

- From the two linear combinations of effects $\frac{1}{2}([i] + [\bar{i}]')$ and $\frac{1}{2}([i] - [i]')$, we get

i	From $\frac{1}{2}([i] + [i]')$	From $\frac{1}{2}([i] - [i]')$
A	$A + CE + FG$	BD
B	$B + CF + EG$	AD
C	$C + AE + BF$	DG
D	D	$AB + CG + EF$
E	$E + AC + BG$	DF
F	$F + BC + AG$	DE
G	$G + BE + AF$	CD

- Thus, we have isolated the main effect of D and all of its two-factor interactions.
- In general, if we add to a fractional design of resolution III or higher a further fraction with the signs of a single factor reversed, then the combined design will provide estimates of the main effect of that factor and its two-factor interactions. This is sometimes called a single-factor fold over.
- Now suppose we add to a resolution III fractional a second fraction in which the signs for all the factors are reversed.
- This type of fold over (sometimes called a full fold over or a reflection) breaks the alias links between all main effects and their two-factor interactions.

EXAMPLE 8.7

A human performance analyst is conducting an experiment to study eye focus time and has built an apparatus in which several factors can be controlled during the test. The factors he initially regards as important are acuity or sharpness of vision (A), distance from target to eye (B), target shape (C), illumination level (D), target size (E), target density (F), and subject (G). Two levels of each factor are considered. He suspects that only a few of these seven factors are of major importance and that high-order interactions between the factors can be neglected. On the basis of this assumption, the analyst decides to run a screening experiment to identify the most important factors and then to concentrate further study on those. To screen these seven factors, he runs the treatment combinations from the 2_{III}^{7-4} design in Table 8.19 in random order, obtaining the focus times in milliseconds, as shown in Table 8.21.

■ **TABLE 8.21**

A 2_{III}^{7-4} Design for the Eye Focus Time Experiment

	Basic Design							
Run	A	B	C	$D = AB$	$E = AC$	$F = BC$	$G = ABC$	**Time**
1	−	−	−	+	+	+	−	*def* 85.5
2	+	−	−	−	−	+	+	*afg* 75.1
3	−	+	−	−	+	−	+	*beg* 93.2
4	+	+	−	+	−	−	−	*abd* 145.4
5	−	−	+	+	−	−	+	*cdg* 83.7
6	+	−	+	−	+	−	−	*ace* 77.6
7	−	+	+	−	−	+	−	*bcf* 95.0
8	+	+	+	+	+	+	+	*abcdefg* 141.8

Seven main effects and their aliases may be estimated from these data. From Equation 8.2, we see that the effects and their aliases are

$$[A] = 20.63 \rightarrow A + BD + CE + FG$$

$$[B] = 38.38 \rightarrow B + AD + CF + EG$$

$$[C] = -0.28 \rightarrow C + AE + BF + DG$$

$$[D] = 28.88 \rightarrow D + AB + CG + EF$$

$$[E] = -0.28 \rightarrow E + AC + BG + DF$$

$$[F] = -0.63 \rightarrow F + BC + AG + DE$$

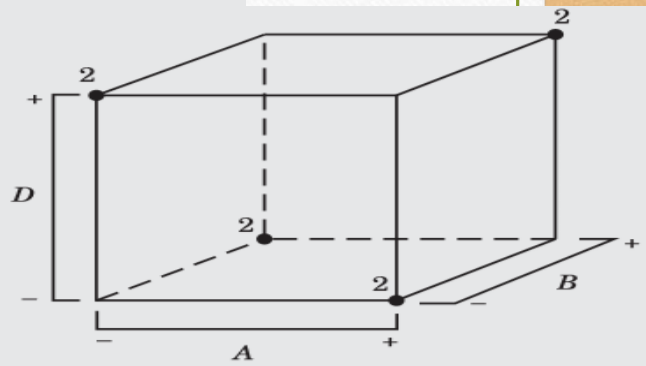
$$[G] = -2.43 \rightarrow G + CD + BE + AF$$

For example, the estimate of the main effect of A and its aliases is

$$[A] = \tfrac{1}{4}(-85.5 + 75.1 - 93.2 + 145.4 - 83.7 + 77.6 - 95.0 + 141.8) = 20.63$$

The three largest effects are $[A]$, $[B]$, and $[D]$. The simplest interpretation of the results of this experiment is that the main effects of A, B, and D are all significant. However, this interpretation is not unique, because one could also logically conclude that A, B, and the AB interaction, or perhaps B, D, and the BD interaction, or perhaps A, D, and the AD interaction are the true effects.

Notice that ABD is a word in the defining relation for this design. Therefore, this 2^{7-4}_{III} design does not project into a full 2^3 factorial in ABD; instead, it projects into two replicates of a 2^{3-1} design, as shown in Figure 8.23. Because the 2^{3-1} design is a resolution III design, A will be aliased with BD, B will be aliased with AD, and D will be aliased with AB, so the interactions cannot be separated from the main effects. The experimenter here may have been unlucky. If he had assigned the factor illumination level to C instead of D, the design would have projected into a full 2^3 design, and the interpretation could have been simpler.



■ **FIGURE 8.23** The 2^{7-4}_{III} design projected into two replicates of a 2^{3-1}_{III} design in A, B, and D

To separate the main effects and the two-factor interactions, the full fold-over technique is used, and a second fraction is run with all the signs reversed. This fold-over design is shown in Table 8.22 along with the observed responses. Notice that when we construct a full fold over of a resolution III design, we (in effect) change the signs on the generators that have an odd number of letters. The effects estimated by this fraction are

$$
\begin{aligned}
[A]' &= -17.68 \rightarrow A - BD - CE - FG \\
[B]' &= 37.73 \rightarrow B - AD - CF - EG \\
[C]' &= -3.33 \rightarrow C - AE - BF - DG \\
[D]' &= 29.88 \rightarrow D - AB - CG - EF \\
[E]' &= 0.53 \rightarrow E - AC - BG - DF \\
[F]' &= 1.63 \rightarrow F - BC - AG - DE \\
[G]' &= 2.68 \rightarrow G - CD - BE - AF
\end{aligned}
$$

- Table 8.22 is in next slide

By combining this second fraction with the original one, we obtain the following estimates of the effects:

i	From $\frac{1}{2}([i] + [i]')$	From $\frac{1}{2}([i] - [i]')$
A	$A = 1.48$	$BD + CE + FG = 19.15$
B	$B = 38.05$	$AD + CF + EG = 0.33$
C	$C = -1.80$	$AE + BF + DG = 1.53$
D	$D = 29.38$	$AB + CG + EF = -0.50$
E	$E = 0.13$	$AC + BG + DF = -0.40$
F	$F = 0.50$	$BC + AG + DE = -1.13$
G	$G = 0.13$	$CD + BE + AF = -2.55$

The two largest effects are B and D. Furthermore, the third largest effect is $BD + CE + FG$, so it seems reasonable to attribute this to the BD interaction. The experimenter used the two factors distance (B) and illumination level (D) in subsequent experiments with the other factors A, C, E, and F at standard settings and verified the results obtained here. He decided to use subjects as blocks in these new experiments rather than ignore a potential subject effect because several different subjects had to be used to complete the experiment.

■ **TABLE 8.22**

A Fold-Over 2_{III}^{7-4} Design for the Eye Focus Experiment

	Basic Design								
Run	***A***	***B***	***C***	***D* = −*AB***	***E* = −*AC***	***F* = −*BC***	***G* = *ABC***		**Time**
1	+	+	+	−	−	−	+	*abcg*	91.3
2	−	+	+	+	+	−	−	*bcde*	136.7
3	+	−	+	+	−	+	−	*acdf*	82.4
4	−	−	+	−	+	+	+	*cefg*	73.4
5	+	+	−	−	+	+	−	*abef*	94.1
6	−	+	−	+	−	+	+	*bdfg*	143.8
7	+	−	−	+	+	−	+	*adeg*	87.3
8	−	−	−	−	−	−	−	(1)	71.9

The Defining Relation for a Fold-Over Design.

- It is often of interest to know the defining relation for the combined design.
- Each separate fraction will have L + U words used as generators: L words of like sign and U words of unlike sign.
- The combined design will have L + U - 1 words used as generators. These will be the L words of like sign and the U - 1 words consisting of independent even products of the words of unlike sign. (Even products are words taken two at a time, four at a time, and so forth.)

- Consider example 8.7
- For the first fraction, the generators are
- I = ABD, I = ACE, I = BCF, and I = ABCG
- and for the second fraction, they are
- I =-ABD, I =-ACE, I =-BCF, and I = ABCG
- In the second fraction we have switched the signs on the generators with an odd number of letters. Also, notice that L + U = 1 + 3 = 4. The combined design will have I = ABCG (the like sign word) as a generator and two words that are independent even products of the words of unlike sign.
- For example, take I = ABD and I = ACE; then I = (ABD)(ACE) = BCDE is a generator of the combined design.

Resolution IV Designs

- A 2^{k-p} fractional factorial design is of resolution IV if the main effects are clear of two factor interactions and some two-factor interactions are aliased with each other.
- Thus, if three factor and higher interactions are suppressed, the main effects may be estimated directly in a 2_{IV}^{k-p} design.
- Resolution IV designs are used extensively as screening experiments
- Any 2_{IV}^{k-p} design must contain at least 2k runs. Resolution IV designs that contain exactly 2k runs are called minimal designs. Resolution IV designs may be obtained from resolution III designs by the process of fold over.

- To fold over, we add to the original fraction a second fraction with all the signs reversed.
- Then the plus signs in the identity column I in the first fraction could be switched in the second fraction, and a (k +1)st factor could be associated with this column.
- The result is a 2_{IV}^{k+1-p} design.
- We look at an example in the next slide

A 2_{IV}^{4-1} Design Obtained by Fold Over

D			
I	*A*	*B*	*C*
Original 2_{III}^{3-1} $I = ABC$			
+	−	−	+
+	+	−	−
+	−	+	−
+	+	+	+
Second 2_{III}^{3-1} with Signs Switched			
−	+	+	−
−	−	+	+
−	+	−	+
−	−	−	−

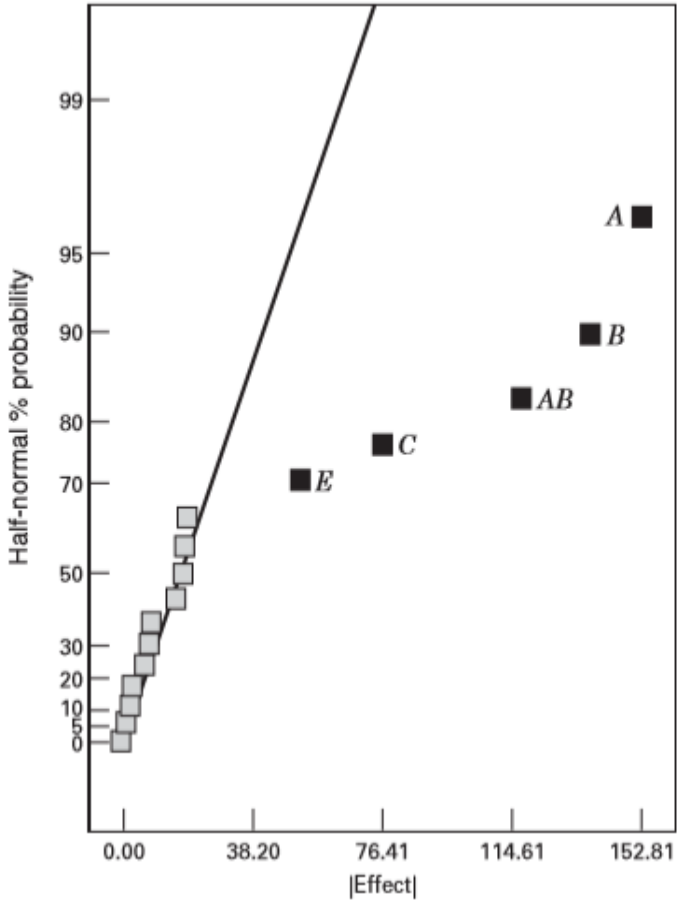
Sequential Experimentation with Resolution IV Designs

- The simplest way to fold over a resolution IV design is to switch the signs on a single variable of the original design matrix.
- This single-factor fold over allows all the two-factor interactions involving the factor whose signs are switched to be separated and accomplishes the (breaking the two-factor interaction aliases involving a specific factor.)
- We consider an example next

- experiment was conducted to study the effects of six factors on the thickness of photoresist coating applied to a silicon wafer.
- The design factors are A =spin speed, B = acceleration, C = volume of resist applied, D = spin time, E = resist viscosity, and F = exhaust rate
- It can be observed (from table in next slide) that the largest main effects are A, B, C, and E, and since these effects are aliased with three-factor or higher interactions, it is logical to assume that these are real effects. However, the effect estimate for the AB CE alias chain is also large. Unless other process knowledge or engineering information is available, we do not know whether this is AB, CE, or both of the interaction effects.

The Initial 2_{IV}^{6-2} Design for the Spin Coater Experiment

A	*B*	*C*	*D*	*E*	*F*	
Speed RPM	Acceleration	Vol (cc)	Time (sec)	Resist Viscosity	Exhaust Rate	Thickness (mil)
−	−	−	−	−	−	4524
+	−	−	−	+	−	4657
−	+	−	−	+	+	4293
+	+	−	−	−	+	4516
−	−	+	−	+	+	4508
+	−	+	−	−	+	4432
−	+	+	−	−	−	4197
+	+	+	−	+	−	4515
−	−	−	+	−	+	4521
+	−	−	+	+	+	4610
−	+	−	+	+	−	4295
+	+	−	+	−	−	4560
−	−	+	+	+	−	4487
+	−	+	+	−	−	4485
−	+	+	+	−	+	4195
+	+	+	+	+	+	4510



■ **FIGURE 8.25** Half-normal plot of effects for the initial spin coater experiment in Table 8.31

- The fold-over design is constructed by setting up a new 2_{IV}^{6-2} fractional factorial design and changing the signs on factor A
- The effects that are estimated from the combined set of runs are (ignoring interactions involving three or more factors)

$$
\begin{aligned}
[A] &= A + BCE + BDF + CDEF & [AE] &= E + ABC + ADF + BCDEF \\
[B] &= B + ACE + ADF + CDEF & [AF] &= F + ABD + ACE + BCDEF \\
[C] &= C + ABE + ADF + BDEF & [BC] &= BC + DF + AE + ABDEF \\
[D] &= D + ABF + ACE + BCEF & [BD] &= BD + CF + AE + ABCEF \\
[E] &= E + ABC + ADF + BCDEF & [BE] &= BE + AC + ABDEF + CDEF \\
[F] &= F + ABD + ACE + BCDEF & [BF] &= BF + CD + ABCEF + ADEF \\
[AB] &= AB + CE + ACDF + BDEF & [CE] &= CE + AB + ACDEF + BDEF \\
[AC] &= AC + BE + ABDF + CDEF & [DE] &= DE + ABCF + ACDEF + BDEF \\
[AD] &= AD + BF + ABCE + CDEF & [EF] &= EF + ABCD + ACDEF + BDEF
\end{aligned}
$$

Block 1- initial run, Block 2-folded run

■ TABLE 8.32

The Completed Fold Over for the Spin Coater Experiment

		A	*B*	*C*	*D*	*E*	*F*	
Std. Order	Block	Speed (RPM)	Acceleration	Vol (cc)	Time (sec)	Resist Viscosity	Exhaust rate	Thickness (mil)
1	1	−	−	−	−	−	−	4524
2	1	+	−	−	−	+	−	4657
3	1	−	+	−	−	+	+	4293
4	1	+	+	−	−	−	+	4516
5	1	−	−	+	−	+	+	4508
6	1	+	−	+	−	−	+	4432
7	1	−	+	+	−	−	−	4197
8	1	+	+	+	−	+	−	4515
9	1	−	−	−	+	−	+	4521
10	1	+	−	−	+	+	+	4610
11	1	−	+	−	+	+	−	4295
12	1	+	+	−	+	−	−	4560
13	1	−	−	+	+	+	−	4487
14	1	+	−	+	+	−	−	4485
15	1	−	+	+	+	−	+	4195
16	1	+	+	+	+	+	+	4510
17	2	+	−	−	−	−	−	4615
18	2	−	−	−	−	+	−	4445
19	2	+	+	−	−	+	+	4475
20	2	−	+	−	−	−	+	4285
21	2	+	−	+	−	+	+	4610
22	2	−	−	+	−	−	+	4325
23	2	+	+	+	−	−	−	4330
24	2	−	+	+	−	+	−	4425
25	2	+	−	−	+	−	+	4655
26	2	−	−	−	+	+	+	4525
27	2	+	+	−	+	+	−	4485
28	2	−	+	−	+	−	−	4310
29	2	+	−	+	+	+	−	4620
30	2	−	−	+	+	−	−	4335
31	2	+	+	+	+	−	+	4345
32	2	−	+	+	+	+	+	4305

- The generators for the original fractions are E ABC and F BCD, and because we changed the signs in column A to create the fold over, the generators for the second group of 16 runs are E ABC and F BCD.
- Since there is only one word of like sign (L 1, U 1) and the combined design has only one generator (it is a one-half fraction), the generator for the combined design is F BCD.
- Furthermore, since ABCE is positive in block 1 and ABCE is negative in block 2, ABCE plus its alias ADEF are confounded with blocks.

- Here we used 16 additional runs but only gained five additional degrees of freedom for estimating two factor interactions. This is not a terribly efficient use of experimental resources. One more efficient alternative is partial fold.
- In a partial fold over (or semifold) we make only half of the runs required for a complete fold over, which for the spin coater experiment would be eight runs.
- The following steps will produce a partial fold-over design:
  - Construct a single-factor fold over from the original design in the usual way by changing the signs on a factor that is involved in a two-factor interaction of interest.
  - Select only half of the fold-over runs by choosing those runs where the chosen factor is either at its high or low level. Selecting the level that you believe will generate the most desirable response is usually a good idea.

Partial fold considering the previous example

The Partial Fold Over for the Spin Coater Experiment

		A	*B*	*C*	*D*	*E*	*F*	
Std. Order	**Block**	**Speed (RPM)**	**Acceleration**	**Vol (cc)**	**Time (sec)**	**Resist Viscosity**	**Exhaust rate**	**Thickness (mil)**
1	1	−	−	−	−	−	−	4524
2	1	+	−	−	−	+	−	4657
3	1	−	+	−	−	+	+	4293
4	1	+	+	−	−	−	+	4516
5	1	−	−	+	−	+	+	4508
6	1	+	−	+	−	−	+	4432
7	1	−	+	+	−	−	−	4197
8	1	+	+	+	−	+	−	4515
9	1	−	−	−	+	−	+	4521
10	1	+	−	−	+	+	+	4610
11	1	−	+	−	+	+	−	4295
12	1	+	+	−	+	−	−	4560
13	1	−	−	+	+	+	−	4487
14	1	+	−	+	+	−	−	4485
15	1	−	+	+	+	−	+	4195
16	1	+	+	+	+	+	+	4510
17	2	−	−	−	−	+	−	4445
18	2	−	+	−	−	−	+	4285
19	2	−	−	+	−	−	+	4325
20	2	−	+	+	−	+	−	4425
21	2	−	−	−	+	+	+	4525
22	2	−	+	−	+	−	−	4310
23	2	−	−	+	+	−	−	4335
24	2	−	+	+	+	+	+	4305

The alias relations from the partial fold over (ignoring interactions involving three or more factors) are

$$
\begin{aligned}
[A] &= A & [AE] &= AE \\
[B] &= B & [AF] &= AF \\
[C] &= C & [BC] &= BC + DF \\
[D] &= D & [BD] &= BD + CF \\
[E] &= E & [BE] &= BE \\
[F] &= F & [BF] &= BF + CD \\
[AB] &= AB & [CE] &= CE \\
[AC] &= AC & [DE] &= DE \\
[AD] &= AD & [EF] &= EF
\end{aligned}
$$

The partial fold-over technique is very useful with resolution IV designs and usually leads to an efficient use of experimental resources. Resolution IV designs always provide good estimates of main effects (assuming that three-factor interactions are negligible), and usually the number of possible two-factor interaction that need to be de-aliased is not large. A partial fold over of a resolution IV design will usually support estimation of as many two-factor interactions as a full fold over. One disadvantage of the partial fold over is that it is not orthogonal. This causes parameter estimates to be correlated and leads to inflation in the standard errors of the effects or regression model coefficients.

Resolution V Designs

- Resolution V designs are fractional factorials in which the main effects and the two-factor interactions do not have other main effects and two-factor interactions as their aliases.
- Consequently, these are very powerful designs, allowing unique estimation of all main effects and two-factor interactions, provided of course that all interactions involving three or more factors are negligible.
- The shortest word in the defining relation of a resolution V design must have five letters.
- The 2^{5-1} design with I = ABCDE is perhaps the most widely used resolutionV design, permitting study of five factors and estimation of all five main effects and all 10 two factor interactions in only 16 runs

Supersaturated Design

- A saturated design is defined as a fractional factorial in which the number of factors or design variables $k = N - 1$, where N is the number of runs.
- In recent years, considerable interest has been shown in developing and using supersaturated designs for factor screening experiments. In a supersaturated design, the number of variables $k>N - 1$, and usually these designs contain quite a few more variables than runs.
- Most design construction techniques are limited computer search procedures based on simple heuristics
  - Others have proposed methods based on optimal design construction techniques